Seven Septoku Theorems

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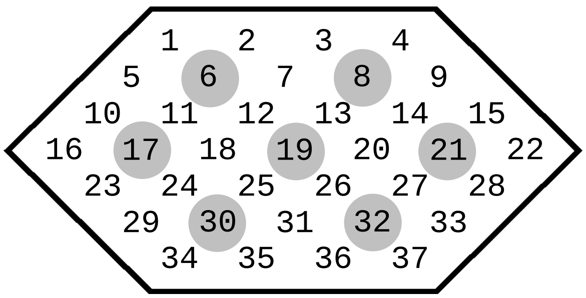
January 4th, 2008

The game of Septoku is a Sudoku variant played on a grid of hexagons. The puzzle was invented in 2006 by Bruce Oberg (http://www.oberg.org). It is played on a hexagonal board of side 4 and 37 spaces as shown below. We label the spaces on the board by numbering them sequentially,



The puzzle is to fill the board with seven symbols (the numbers 1 to 7) such that

1) Each row (in all three directions) contains different numbers. There are seven rows in each direction, so 21 rows in all.
2) The seven circular regions (with gray centers) must each contain all seven symbols. For example the spaces {1, 2, 5, 6, 7, 11, 12} must contain all of the numbers 1 to 7.

We will refer to a circular region by its center, i.e. "circle 6" refers to the spaces {1, 2, 5, 6, 7, 11, 12}.

As with Sudoku, it is now possible to create puzzles using a blank board with a number of **clues** (or **seeds**) filled in. The puzzle is to fill out the rest of the board so the above rules are satisfied. A well-crafted puzzle has only one solution.

Unlike Sudoku, a valid Septoku board must also satisfy certain additional constraints that are not at all obvious. In fact, we will see that the above rules above are so stringent that there are only 6 valid boards, up to reflections and rotations of the board, and permutations of the seven symbols.

Theorem 1: In a valid Septoku board, each circle center {6,8,17,19,21,30,32} must have a unique symbol.

Proof: Suppose otherwise. Then there must be some symbol that occurs twice among the circle centers. The center symbol is clearly unique by the rules of the puzzle, so without loss of generality, assume a 1 appears in spaces 6 and 21. There must be a 1 in the center circle 19, and this can only be at space 13 or 25. But if it is at 25, then circle 8 can contain no 1 at all. Thus there must be a 1 in space 13. There must be a 1 in circle 17, and this can now only be at space 24, and a 1 in circle 32, which can only be at space 31. But now this is not a valid Septoku board, because circle 30 contains two 1's.

<u>Theorem 2</u>: In a valid Septoku board, each corner {1,4,16,22,34,37} plus the center {19} must have a unique symbol.

Proof: Suppose otherwise. Then there must be some symbol that occurs twice among the corners (the center symbol is clearly unique by the rules of the puzzle). Without loss of generality, assume 1 occurs at spaces 1 and 22. There must be a 1 in the center circle 19, and this can only be at space 13 or 25. In either case, no circle center can now contain a 1 without introducing two 1's in the same row. This contradicts Theorem 1, which implies that there must be a 1 in some circle center.

<u>Theorem 3</u>: In a valid Septoku board, each symbol must appear at least five times.

Proof: Suppose the symbol 1 appears four or fewer times. By the Theorems 1 and 2 each symbol must appear in a circle center, and in a corner (or once in the center, 19). If 1 occurs in a corner and one circle center, then this covers only two of the circles out of seven. There is no way the remaining two 1's can cover the remaining five circles, since one space is in common with at most two circles. The remaining possibility is that the 1 is in the center, space 19. However it is not possible for the three remaining 1's to cover the six remaining circles without being placed in the same row.

If each symbol occurs 5 times this covers only 35 of the 37 board locations, so *at least one symbol must occur more than 5 times*. In fact there are only two possibilities:

1. Six of the symbols appear 5 times, one symbol appears 7 times.
2. Five of the symbols appear 5 times, two symbols appear 6 times.

We shall now see that both of these possibilities can occur.

<u>Theorem 4</u>: Not counting reflections, rotations, and symbol permutations, there are only **six** valid Septoku boards.

Proof: By applying a symbol permutation, any valid board has a version where the center looks like this:

```
 1 2
6 7 3
 5 4
```

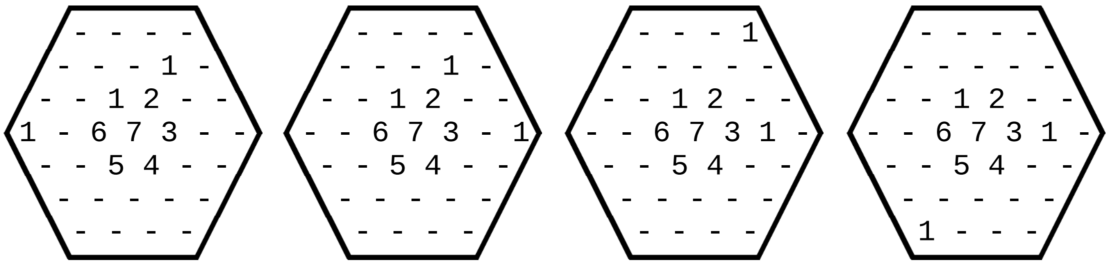
The reason why this configuration was chosen is that the symbol permutation (123456) rotates the pattern in the center circle counter-clockwise by 60 degrees, and the permutation (654321) does a clockwise rotation[1]. A solution with this pattern in the center is said to be in **standard form**.

From Theorem 1 we know that the symbol 1 must appear in some circle center. It suffices to consider only the cases where this 1 is in space 8 or 21. For if the 1 is at 17 or

---

[1] We use standard notation to represent a permutation as a product of cycles. The cycle (123456) means that the permutation takes 1 to 2, 2 to 3, 3 to 4, 4 to 5, 5 to 6, 6 to 1 and leaves 7 alone.

30 we reflect the board across the diagonal line formed by the spaces {1, 6, 12, 19, 26, 32, 37}, and then apply the permutation (26)(35) to bring it into standard form with a 1 at space 8 or 21. From Theorem 2 we know that the symbol 1 must appear in some corner, and thus all possible Septoku boards have a standard form that looks like one of these four:



Puzzle 1 Puzzle 2 Puzzle 3 Puzzle 4

Therefore, if we find ALL solutions to the four puzzles above, we will find all possible Septoku boards. This may seem difficult, but actually there are not many solutions.

Proposition 1: Puzzle 4 is unsolvable.

Proof: Circle 17 must contain a 1, but there is no space where a 1 can be placed without producing two 1's in some row.

Proposition 2: Puzzles 1-3 each have 4 solutions.

Proof: It may be possible to prove this analytically, but for now we use an integer programming model to make sure we have found all solutions.

We define an unknown binary 37x7 matrix X as:

$$X(M,N)=1 \text{ if space M is labeled N, 0 otherwise}$$

X must satisfy the constraints:

For all spaces M, $\sum_N X(M,N)=1$	Every space must have exactly one label.
For each region $R_i \subset M$ and each label N, $\sum_{R_i} X(R_i,N)\le 1$	Every region contains no duplicate labels.
For each clue c, $X(M_c,N_c)=1$	The starting clues or seeds are satisfied.

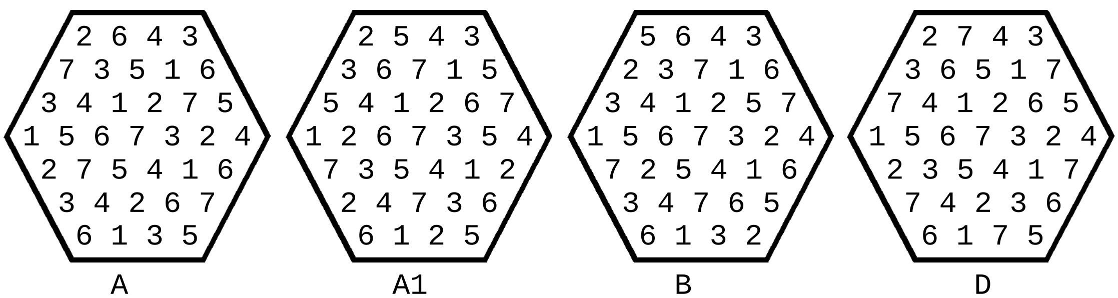
There are several programming languages for solving such problems, we use GAMS (General Algebraic Modeling System). The details of the model are in the GAMS input file in Appendix A.

A specialized solver is needed to solve integer programming problems. Fortunately, a free solver is available on the web, the NEOS server. You can submit the GAMS models in Appendix A using the web form at the link below. The model will be solved (in under 1 second) and the results will be emailed to you. The submission link is:

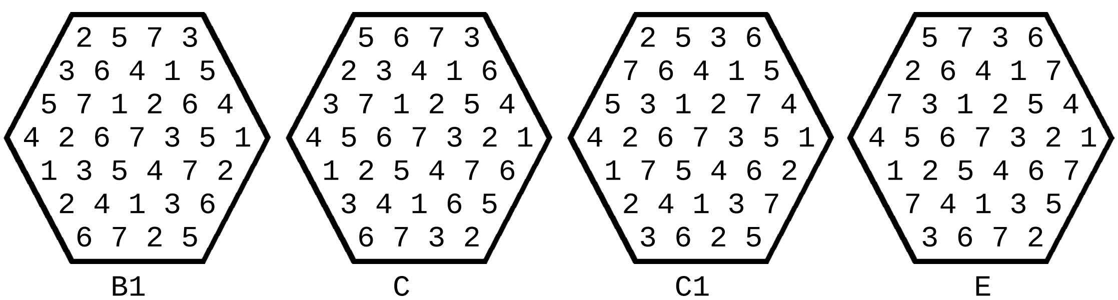
http://neos.mcs.anl.gov/neos/solvers/milp:XpressMP/GAMS.html

The solver will return a single solution, if one exists. To get additional solutions, we include another constraint that eliminates only the solution that has been found. After doing this four times, the solver reports that no more solutions exist. For the four puzzles, the solutions returned by the GAMS solves are:

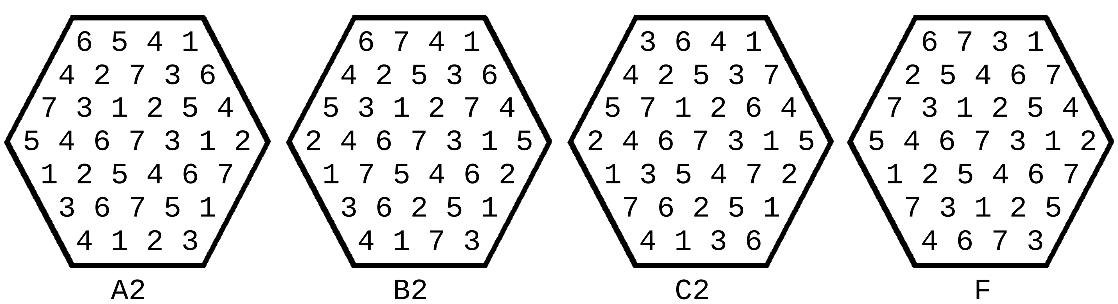
Puzzle #1 Solutions:



Puzzle #2 Solutions:



Puzzle #3 Solutions:



Since there are 12 solutions to puzzles #1-#4, one might think there are exactly 12 Septoku boards. However, by applying rotations, reflections, and permutations, we can show that some boards can be converted into others. We define:

1. (Rot) as the transformation that rotates the board clockwise 60 degrees.

   Therefore $(Rot)^{-1}$ is a rotation counter-clockwise by 60 degrees.

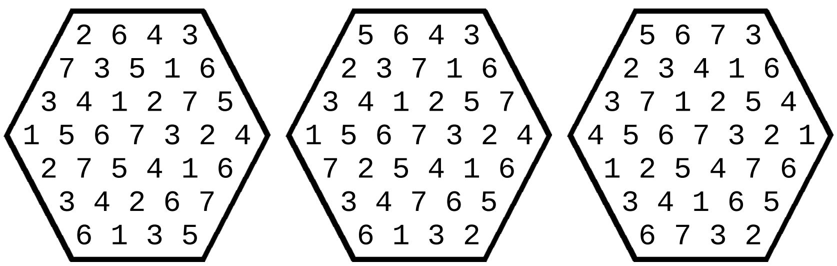
2. (Flx) as the reflection of the board across the *x*-axis.

Then we have the following identities between boards:

$$A(Rot)^{-1}(654321) = A1 \qquad A(Flx)(Rot)(16)(25)(34) = A2$$
$$B(Rot)^{-1}(654321) = B1 \qquad B(Flx)(Rot)(16)(25)(34) = B2$$
$$C(Rot)^{-1}(654321) = C1 \qquad C(Flx)(Rot)(16)25)(34) = C2$$

Thus, there are only six boards. Three of them contain five of each number except for 3 and 6, of which there are six:

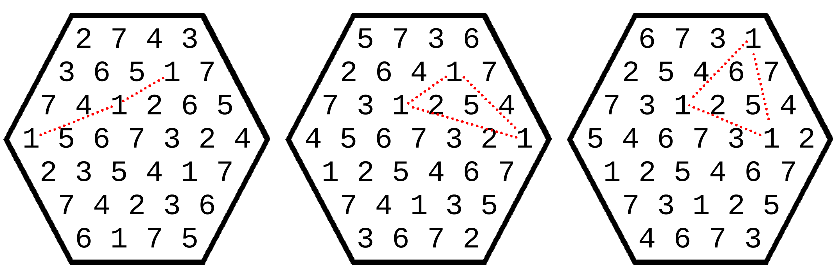
Boards A, B, C



A B C

Then the other three boards have five of each number, except for the central number 7, of which there are seven. The number 7 appears in each of the 21 rows.

Boards D, E, F



D E F

Finally, how do we know that none of the six boards above can be transformed into one another by applying reflections, rotations, and permutations (which would mean there are fewer than 6 solutions)? For example, how do we know that we can't apply some symmetry transformation (Sym) followed by a permutation *P* to convert board *A* into *B*?

$$A(Sym)(P) = B$$

We note that no symmetry transformation changes the value of the center, so for the above equation to hold, *P* must leave 7's alone. Therefore, $(Sym)(P)$ must map 7's in one board to 7's in the other. It is impossible for any board in {A, B, C} to be transformed into one of {D, E, F}, as they contain different numbers of 7's.

As far as transforming *A* to *B*, there are only two rotations of *A* that produce the pattern of 7's in *B*. The central circle must be the same in both boards, this fixes the permutation *P*, and by direct calculation we find that neither of these rotations plus permutation yield *B*. Similarly, it is time consuming but not difficult to verify that **none** of the six boards can be transformed into any of the others.

We also verified this using a computer, applying each of the 12 symmetry transformations, and then applying each of the 7! = 5040 permutations in turn. A program that did this found that there was no way to transform a board of one of the six fundamental types A-F into any of the others (this program also identified A2, A3, B2, B3, C2, C3).

<u>Theorem 5</u>: Counting reflections, rotations, and symbol permutations as separate boards, there are 120,960 valid Septoku boards.

Proof: Each solution board has 7! = 5,040 different versions using all possible permutations of symbols. These boards are all distinct, since each permutation is distinct. We can then apply the 12 symmetry transformations to generate (5,040)(12) = 60,480 board positions. However, not all of these will be distinct. If (Sym) is a symmetry transformation and *P* is a permutation, we need to figure out under what conditions A is unchanged after application of (Sym) followed by *P*, or

$$A(Sym)(P) = A$$

Notice that none of the symmetry transformations change the value of the central hole. Therefore, for the above equation to hold the permutation *P* cannot change the value of the central hole (which is always 7 in standard form), so the symmetry transformation must leave all 7's unchanged.

We note that for the boards *A*, *B*, and *C*, the only symmetry transformation that leaves all 7's unchanged is a rotation of the board by 180 degrees. We can then calculate that

$$A(Rot)^3(14)(25)(36) = A$$

and similarly for *B* and *C*. This effectively halves the number of distinct boards generated from these three, giving us 3(5,040)(12)/2 = 90,720 different solutions.

The 7's in the boards D, E and F are unchanged by a rotation of 60 degrees, and

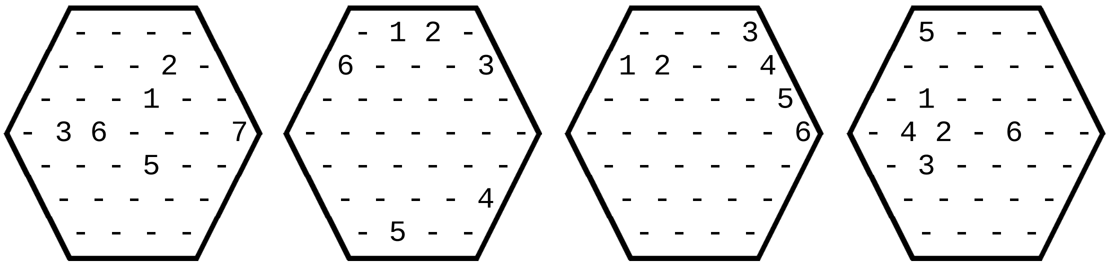
$$D(Rot)(123456) = D$$

and similarly for E and F. This reduces the number of distinct boards generated in these cases by a factor of 6, giving us 3(5,040)(12)/6 = 30,240. The total number of possible boards is therefore 90,720 + 30,240 = 120,960 = 24(7!).

<u>Theorem 6</u>: The smallest number of clues (or seeds) for a Septoku puzzle to have a unique solution is six.

Proof: Clearly the answer cannot be less than six. If this were possible, there must be two of the seven numbers not even among the clues. If there is one solution, a different one can be obtained by swapping these two numbers.

To show the answer is six, we need only find a puzzle with six clues and a unique solution. By the same argument as the previous paragraph, the six clues must be for six of the seven numbers (all different values). We wrote a program that takes one of the six valid boards, and considers all combinations of 6 different clues. The program then checks each such puzzle for a unique solution by matching against all possible symmetry transformations and permutations of all six possible boards. In this manner hundreds of puzzles were generated with six clues and a unique solution. Four of them are shown below:



Working by hand, finding solutions to these puzzles is difficult (although the first is not so bad as we will show below). It is not easy to prove that the solutions are unique. Nonetheless, uniqueness can be established using an integer programming model. An example input file for this is given in Appendix A.

We note that Theorems 1-3 are useful for solving Septoku puzzles by hand. There is one more theorem that is even more powerful.

<u>**Theorem 7**</u>**:** In a valid Septoku board, the symbols can be partitioned into three pairs, with the remaining symbol being the value of the central space 19. Let *x* and *y* be *any* pair of spaces that map to each other under 180 degree rotation (for example spaces 1 and 37, or 2 and 36, or 7 and 31, etc.). Then either

1. *x* and *y* have values equal to one of the three pairs, or
2. *x* and *y* both have the value of the central hole (clearly this is only possible if *x* and *y* are not in the same row).

Proof: It is easy to check that each of the six solution boards A-F satisfies Theorem 7. Since any solution is a symmetry transformation plus permutation of one of them, all solutions satisfy Theorem 7.

For example, in the solution boards A-F in standard form, the pairs are always {1,4}, {2,5} and {3,6}, while the center is always 7. Theorem 7 says that if we find a 1 in some space, *there must be a 4 in the space diametrically opposite*, and similarly for the other pairs. If there is a 7 in some space, there must be another 7 in the space diametrically opposite.

In solving Septoku puzzles, once we figure out what the pairings are, Theorem 7 is very powerful in filling out the board. Once we figure out one board location we can also fill in the space diametrically opposite. This actually makes many puzzles very easy, and the only ones that are not easy are those where it is hard to figure out the pairs.

For example, we now show how Theorem 7 can be used to solve the first 6-clue puzzle. First, it is clear that the center can only be the missing symbol, 4, and the only possibility for space 20 is a 2. The board now looks like (space numbering shown on the right for convenience):

```
   - - - -              1  2  3  4
  - - - 2 -           5  6  7  8  9
 - - - 1 - -        10 11 12 13 14 15
- 3 6 4 2 – 7      16 17 18 19 20 21 22
 - - - 5 - -        23 24 25 26 27 28
  - - - - -           29 30 31 32 33
   - - - -              34 35 36 37
```

Now consider the pairs of Theorem 7. Clearly {6,2} from the center is one such pair, which implies that space 30 must be a 6. 7 must form a pair with either 1 or 5. If {7,5} is a pair, then space 12 is a 7, spaces 4 and 16 must be a 5's and space 34 a 7. However we can then not place a 7 in circle 17. So we now know that the three pairs are {2,6}, {1,7} and {3,5} and the board looks like this:

```
   - - - -
  - - - 2 -
 - - 3 1 - -
1 3 6 4 2 5 7
 - - 7 5 - -
  - 6 - - -
   - - - -
```

Space 6 must now be either a 1 or a 7 (by Theorem 1). If it is a 7 then circle 32 cannot contain a 7, so space 6 must be a 1 and space 32 is a 7. We can now determine that spaces 28 and 31 must be 1's, and spaces 10 and 7 must be 7's.

```
   - - - -
  - 1 7 2 -
 7 - 3 1 - -
1 3 6 4 2 5 7
 - - 7 5 - 1
  - 6 1 7 -
   - - - -
```

Space 34 must be a 3 or a 5, but if it is a 5 then circle 8 can contain no 5. Thus space 34 must be a 3, and space 4 must be a 5. Circle 21 must contain a 3, and it can only be at 27.

```
   - - - 5
  - 1 7 2 -
 7 - 3 1 - -
1 3 6 4 2 5 7
 - - 7 5 3 1
  - 6 1 7 -
   3 - - -
```

Space 1 must be a 2 or a 6, but if it is a 6 then circle 32 can contain no 6. Thus space 1 must be a 2, and space 37 a 6.

```
   2 - - 5
  - 1 7 2 -
 7 - 3 1 - -
1 3 6 4 2 5 7
 - - 7 5 3 1
  - 6 1 7 -
   3 - - 6
```

We can now readily fill in the rest of the solution

```
   2 6 4 5
  4 1 7 2 3
 7 5 3 1 6 4
1 3 6 4 2 5 7
 4 2 7 5 3 1
  5 6 1 7 4
   3 4 2 6
```

In fact, this board *X* is related to board *E*, as can be seen by verifying the identity

$$X\,(Rot)^{-1}(Flx)(Rot)(1625473) = E$$

Conclusions

Septoku is a puzzle where the set of solutions is far more constrained than it first appears from the rules. We have found that there are only six fundamentally different solution boards, and this knowledge has allowed us to create 6-clue puzzles that can be quite difficult to solve by hand. Theorems 1, 2, 3 and 7 are useful for solving Septoku puzzles. However, for the hardest 6-clue puzzles generated, the author is unable to solve them without an integer program!

Solutions

Here we present the unique solution for each of the 6-clue puzzles (clues in red):

2 6 4 5
4 1 7 **2** 3
7 5 3 **1** 6 4
1 **3** **6** 4 2 5 **7**
4 2 7 **5** 3 1
5 6 1 7 4
3 4 2 6

E

4 **1** **2** 5
6 5 7 1 **3**
1 2 3 6 4 7
3 4 5 7 2 6 1
7 6 4 1 5 3
1 3 7 2 **4**
2 **5** 3 6

C

5 7 1 **3**
1 **2** 6 5 **4**
6 3 4 2 7 **5**
2 4 5 7 1 3 **6**
1 7 6 3 4 2
3 1 2 6 5
4 5 7 1

C

5 2 1 6
3 6 4 7 2
6 **1** 7 5 3 4
7 **4** **2** 3 **6** 5 1
5 **3** 4 1 7 2
6 1 5 2 3
2 7 6 4

A

(is equivalent to this board by a symmetry transformation plus permutation)

Appendix A: GAMS model files

```
$TITLE Septoku Board Puzzle #1
* This GAMS program solves Septoku puzzles
* (simular to Sudoku, but played on a hexagonal grid)
* Author: George Bell
* Date: January 2, 2008

$Onlisting
SETS
M spaces  /1*37/
N labels  /1*7/
R regions /1*28/

* The Septoku Board (37 spaces or cells):
*         1   2   3   4
*       5  <6>  7  <8>  9
*    10  11  12  13  14  15
*  16 <17> 18 <19> 20 <21> 22
*    23  24  25  26  27  28
*      29 <30> 31 <32> 33
*        34  35  36  37

* Regions that must contain distinct values
* Regions  1-7  are the (horizontal) rows
* Regions  8-14 are the (up-right slanting) rows
* Regions 15-21 are the (down-right slanting) rows
* Regions 22-28 are the circular regions (with centers <*> in diagram)
* Note that all regions have between 4 and 7 spaces
SET REGIONS (R,M)
     /   1.1,  1.2,  1.3,  1.4,
         2.5,  2.6,  2.7,  2.8,  2.9,
         3.10, 3.11, 3.12, 3.13, 3.14, 3.15,
         4.16, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22,
         5.23, 5.24, 5.25, 5.26, 5.27, 5.28
         6.29, 6.30, 6.31, 6.32, 6.33,
         7.34, 7.35, 7.36, 7.37,
         8.1,  8.5,  8.10, 8.16,
         9.2,  9.6,  9.11, 9.17, 9.23,
        10.3, 10.7, 10.12,10.18,10.24,10.29,
        11.4, 11.8, 11.13,11.19,11.25,11.30,11.34,
        12.9, 12.14,12.20,12.26,12.31,12.35,
        13.15,13.21,13.27,13.32,13.36,
        14.22,14.28,14.33,14.37,
        15.4, 15.9, 15.15,15.22,
        16.3, 16.8, 16.14,16.21,16.28,
        17.2, 17.7, 17.13,17.20,17.27,17.33,
        18.1, 18.6, 18.12,18.19,18.26,18.32,18.37,
        19.5, 19.11,19.18,19.25,19.31,19.36,
        20.10,20.17,20.24,20.30,20.35,
        21.16,21.23,21.29,21.34,
        22.1, 22.2, 22.5, 22.6, 22.7, 22.11,22.12,
        23.3, 23.4, 23.7, 23.8, 23.9, 23.13,23.14,
        24.10,24.11,24.16,24.17,24.18,24.23,24.24,
        25.12,25.13,25.18,25.19,25.20,25.25,25.26,
        26.14,26.15,26.20,26.21,26.22,26.27,26.28,
        27.24,27.25,27.29,27.30,27.31,27.34,27.35,
        28.26,28.27,28.31,28.32,28.33,28.36,28.37/;

* The clues at the start of this puzzle that must be satisfied
SET CLUES (M,N)
```

```
      /12.1, 13.2, 20.3, 26.4, 25.5, 18.6, 19.7, 8.1, 16.1 /;

*         1   2   3   4
*       5  <6>  7  <8>  9
*    10  11  12  13  14  15
*  16 <17> 18 <19> 20 <21> 22
*    23  24  25  26  27  28
*      29 <30> 31 <32> 33
*        34  35  36  37

* Solutions that may or may not be allowed (to find ALL solutions)
SET SOL1 (M,N)
      /  1.2,  2.5,  3.4,  4.3,
         5.3,  6.6,  7.7,  8.1,  9.5,
        10.5, 11.4, 12.1, 13.2, 14.6, 15.7,
        16.1, 17.2, 18.6, 19.7, 20.3, 21.5, 22.4,
        23.7, 24.3, 25.5, 26.4, 27.1, 28.2,
        29.2, 30.4, 31.7, 32.3, 33.6,
        34.6, 35.1, 36.2, 37.5/;
SET SOL2 (M,N)
      /  1.2,  2.6,  3.4,  4.3,
         5.7,  6.3,  7.5,  8.1,  9.6,
        10.3, 11.4, 12.1, 13.2, 14.7, 15.5,
        16.1, 17.5, 18.6, 19.7, 20.3, 21.2, 22.4,
        23.2, 24.7, 25.5, 26.4, 27.1, 28.6,
        29.3, 30.4, 31.2, 32.6, 33.7,
        34.6, 35.1, 36.3, 37.5/;
SET SOL3 (M,N)
      /  1.5,  2.6,  3.4,  4.3,
         5.2,  6.3,  7.7,  8.1,  9.6,
        10.3, 11.4, 12.1, 13.2, 14.5, 15.7,
        16.1, 17.5, 18.6, 19.7, 20.3, 21.2, 22.4,
        23.7, 24.2, 25.5, 26.4, 27.1, 28.6,
        29.3, 30.4, 31.7, 32.6, 33.5,
        34.6, 35.1, 36.3, 37.2/;
SET SOL4 (M,N)
      /  1.2,  2.7,  3.4,  4.3,
         5.3,  6.6,  7.5,  8.1,  9.7,
        10.7, 11.4, 12.1, 13.2, 14.6, 15.5,
        16.1, 17.5, 18.6, 19.7, 20.3, 21.2, 22.4,
        23.2, 24.3, 25.5, 26.4, 27.1, 28.7,
        29.7, 30.4, 31.2, 32.3, 33.6,
        34.6, 35.1, 36.7, 37.5/;

SCALARS ALLOW1 Set to 37 to allow SOL1 and 36 to not allow it /36/
SCALARS ALLOW2 Set to 37 to allow SOL2 and 36 to not allow it /36/
SCALARS ALLOW3 Set to 37 to allow SOL3 and 36 to not allow it /36/
SCALARS ALLOW4 Set to 37 to allow SOL4 and 36 to not allow it /36/;

VARIABLES
X(M,N) Equal to one if space M is labeled N
OBJECT Objective function (unimportant)

BINARY VARIABLES X;

EQUATIONS

OBJ           The sum of all labels (unimportant)
ALLFILLED(M) All spaces must have exactly one entry
REGDIST(R,N) Each region must have distinct entries
PUZZLE        The clues defining the puzzle
UNIQ1         Determines if SOL1 is allowed or not
UNIQ2         Determines if SOL2 is allowed or not
```

```
UNIQ3        Determines if SOL3 is allowed or not
UNIQ4        Determines if SOL4 is allowed or not;

OBJ..
     SUM((M,N),X(M,N)) =E= OBJECT;
ALLFILLED(M)..
     SUM(N,X(M,N)) =E= 1;
REGDIST(R,N)..
     SUM(M$REGIONS(R,M),X(M,N)) =L= 1;
PUZZLE..
     SUM(CLUES,X(CLUES)) =G= CARD(CLUES);
UNIQ1..
     SUM(SOL1,X(SOL1)) =L= ALLOW1;
UNIQ2..
     SUM(SOL2,X(SOL2)) =L= ALLOW2;
UNIQ3..
     SUM(SOL3,X(SOL3)) =L= ALLOW3;
UNIQ4..
     SUM(SOL4,X(SOL4)) =L= ALLOW4;


MODEL SEPTOKU /all/;

OPTION OPTCR=.0001;
OPTION ITERLIM=9000000;
SOLVE SEPTOKU USING MIP MINIMIZING OBJECT;

OPTION X:0:0:1;
DISPLAY X.L;
DISPLAY UNIQ1.L;
DISPLAY UNIQ2.L;
DISPLAY UNIQ3.L;
DISPLAY UNIQ4.L;
```

```
$TITLE Septoku 6 clue problem 2
* This GAMS program solves Septoku puzzles
* (simular to Sudoku, but played on a hexagonal grid)
* Author: George Bell
* Date: January 4, 2008

$Onlisting
SETS
M spaces  /1*37/
N labels  /1*7/
R regions /1*28/

* The Septoku Board (37 spaces or cells):
*          1   2   3   4
*       5  <6>  7  <8>  9
*    10  11  12  13  14  15
*  16 <17> 18 <19> 20 <21> 22
*    23  24  25  26  27  28
*      29 <30> 31 <32> 33
*        34  35  36  37

* Regions that must contain distinct values
* Regions  1-7  are the (horizontal) rows
* Regions  8-14 are the (up-right slanting) rows
* Regions 15-21 are the (down-right slanting) rows
* Regions 22-28 are the circular regions (with centers <*> in diagram)
* Note that all regions have between 4 and 7 spaces
```

```
SET REGIONS (R,M)
     /   1.1,  1.2,  1.3,  1.4,
         2.5,  2.6,  2.7,  2.8,  2.9,
         3.10, 3.11, 3.12, 3.13, 3.14, 3.15,
         4.16, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22,
         5.23, 5.24, 5.25, 5.26, 5.27, 5.28
         6.29, 6.30, 6.31, 6.32, 6.33,
         7.34, 7.35, 7.36, 7.37,
         8.1,  8.5,  8.10, 8.16,
         9.2,  9.6,  9.11, 9.17, 9.23,
        10.3, 10.7, 10.12,10.18,10.24,10.29,
        11.4, 11.8, 11.13,11.19,11.25,11.30,11.34,
        12.9, 12.14,12.20,12.26,12.31,12.35,
        13.15,13.21,13.27,13.32,13.36,
        14.22,14.28,14.33,14.37,
        15.4, 15.9, 15.15,15.22,
        16.3, 16.8, 16.14,16.21,16.28,
        17.2, 17.7, 17.13,17.20,17.27,17.33,
        18.1, 18.6, 18.12,18.19,18.26,18.32,18.37,
        19.5, 19.11,19.18,19.25,19.31,19.36,
        20.10,20.17,20.24,20.30,20.35,
        21.16,21.23,21.29,21.34,
        22.1, 22.2, 22.5, 22.6, 22.7, 22.11,22.12,
        23.3, 23.4, 23.7, 23.8, 23.9, 23.13,23.14,
        24.10,24.11,24.16,24.17,24.18,24.23,24.24,
        25.12,25.13,25.18,25.19,25.20,25.25,25.26,
        26.14,26.15,26.20,26.21,26.22,26.27,26.28,
        27.24,27.25,27.29,27.30,27.31,27.34,27.35,
        28.26,28.27,28.31,28.32,28.33,28.36,28.37/;

* The clues at the start of this puzzle that must be satisfied
SET CLUES (M,N)
      / 2.1, 3.2, 9.3, 33.4, 35.5, 5.6 /;

* A solution that may not be allowed (to find uniqueness)
SET SOL1 (M,N)
      /  1.4,  2.1,  3.2,  4.5,
         5.6,  6.5,  7.7,  8.1,  9.3,
        10.1, 11.2, 12.3, 13.6, 14.4, 15.7,
        16.3, 17.4, 18.5, 19.7, 20.2, 21.6, 22.1,
        23.7, 24.6, 25.4, 26.1, 27.5, 28.3,
        29.1, 30.3, 31.7, 32.2, 33.4,
        34.2, 35.5, 36.3, 37.6/;

SCALARS ALLOW1 Set to 37 to allow SOL1 and 36 to not allow it /36/;

VARIABLES
X(M,N) Equal to one if space M is labeled N
OBJECT Objective function (unimportant)

BINARY VARIABLES X;

EQUATIONS

OBJ          The sum of all labels (unimportant)
ALLFILLED(M) All spaces must have exactly one entry
REGDIST(R,N) Each region must have distinct entries
PUZZLE       The clues defining the puzzle
UNIQ1        Determines if SOL1 is allowed or not;

OBJ..
     SUM((M,N),X(M,N)) =E= OBJECT;
ALLFILLED(M)..
```

```
     SUM(N,X(M,N)) =E= 1;
REGDIST(R,N)..
     SUM(M$REGIONS(R,M),X(M,N)) =L= 1;
PUZZLE..
     SUM(CLUES,X(CLUES)) =G= CARD(CLUES);
UNIQ1..
     SUM(SOL1,X(SOL1)) =L= ALLOW1;

MODEL SEPTOKU /all/;

OPTION OPTCR=.0001;
OPTION ITERLIM=9000000;
SOLVE SEPTOKU USING MIP MINIMIZING OBJECT;

OPTION X:0:0:1;
DISPLAY X.L;
DISPLAY UNIQ1.L;
```